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by

F.B. Smith

(Meteorological Office)

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SUMMARY

Solutions are obtained for the mean concentrations of an emittant dispersed from a cross-wind line source into a turbulent layer of finite depth. Turbulent mixing, represented qualitatively by an eddy diffusivity coefficient K(z), is one of the dominant causes of the dispersion. Unlike most previous solutions based on turbulent diffusion alone, the present ones are derived using diffusivity coefficients all of which fall to zero, either smoothly or sharply, at the top of the finite turbulent layer (as, for example, might happen at the base of a strong inversion).

A further important dispersion effect which also may occur is that of convective transfer. An idealized model, retaining only the dominant features of the convection process, is proposed. The natural diffusion process is augmented by the transport of the emittant, at ground level concentrations, to the top of the layer by thermally generated bubbles or streams rising from the ground. This is in contrast to the conventional method in which the convective transfer is represented by an enhanced value of K(z). In the present model, the convection is balanced by a uniform subsidence throughout the layer. It is assumed that, as the bubbles or streams rise, no mixing takes place with the surrounding atmosphere, that the concept of an eddy diffusivity coefficient can be retained to represent the dispersive processes other than that of free convection and that this coefficient, as without convection, falls smoothly to zero at the inter-surface between the turbulent and stable layers.

The method used in these solutions is that of Heaviside Calculus. One specific case of the convection - diffusion model is evaluated, and is compared with the solution of equal K(z) when convection is absent. The important role of convection is demonstrated.

I. INTRODUCTION TO THE SIMPLE DIFFUSION MODEL

Most of the present formulae for the vertical distribution of concentration of non-buoyant matter emitted from sources and dispersed by simple diffusion are likely to fail at moderately large distances downstream of the source because of variations aloft in the functional form of the eddy diffusivity coefficient, K(z), not accounted for in the initial assumptions. This is particularly true in cases when a turbulent layer adjacent to the ground has above it a stable layer in which turbulent mixing is virtually non-existent. The diffusion will then take place in a region of limited depth and be equivalent to diffusion between parallel planes in which the eddy diffusivity coefficient may take a variety of forms. These coefficients will be characterized by falling to zero at the upper plane.

The height of the inter-surface between two such layers introduces a length scale into the problem, otherwise absent, which not only implies that the concentration will eventually become uniform in the lower layer, but also that dimensional arguments could give a measure of the distance X downstream at which this uniformity is achieved.

A further phenomenon which can lead to important differences in the distribution of concentration is that of convection. The convection concerned will not be the kind arising out of the inherent buoyancy in hot plumes relative to their environment, but rather from the usual type of convection arising from solar heating of the ground and, as such, will be independent of the source. It is clearly desirable to make some attempt to represent statistically the effect of free convection in transporting matter; in particular in its effect in taking the emittant from a layer near the ground up to the inter-surface between the

turbulent and stable layers. This effect will be in addition to the normal diffusive processes. The nature and effect of convection will be further dealt with in II.

The object of this paper is to account for the above; namely, the limitation in depth of the layer through which the emittant will be distributed and the augmentative effect of free convection on the rate at which the emittant is spread throughout the layer. The approach will be to produce analytic results by attempting to make a compromise between the complex physical phenomena and the mathematics.

The effect of free convection will most readily be appreciated by comparison with the corresponding result when convection is absent but in which all other factors are the same. A comparison between two such models is made in VII. The paper will tend to divide itself into two parts; a discussion of the problem of simple diffusion between parallel planes (without convection) for several functional forms of the eddy diffusivity coefficient and a more restricted examination of the diffusion-convection problem in which the dominant features of the process are described. What remains of this section and III, IV and V will be devoted to the former type problem and II and VI to the latter. These parts will be linked by the comparison in VII.

In the case of simple diffusion, the mathematical problem consists of finding the solution of the diffusion equation with variable eddy diffusivity coefficient $K(z)$ and mean wind velocity $u(z)$, between two parallel planes across which there can be no transport of the emittant. These two planes represent the ground and the inter-surface. The source of the emittant is elevated at a height h (which may be zero). Where exact solutions cannot be determined, asymptotic solutions for small distances downstream (compared with X) will be of interest.

On the assumption that $u(z)$ and $K(z)$ may be represented as powers of the height (measured from any prescribed origin), a transformation of the independent variables x and z (Sutton 1943) changes the diffusion equation to one which represents conditions in an atmosphere of constant wind velocity. Hence, in this paper, the wind velocity will without loss of generality be considered constant with height.

Moreover we shall restrict ourselves to the problem of a continuous infinite line source, not only because in present-day application this is of primary interest but also because the point source solutions may be derived as an extension of the present solutions by the method used in an earlier paper (Smith 1957).

Two alternative approaches have been suggested for the solution of the two-dimensional mathematical problem. They are in many ways very similar, but in actual handling each offers its own particular difficulties. However as far as obtaining actual results is concerned, where these exist, the two methods are equally effective; where a solution can be found by one, it is invariably found that the other will also produce the result. The first method suggested by Sutton (1943) consists of obtaining a set of fundamental solutions involving a parameter; this parameter being restricted to a sequence of eigenvalues determined by the boundary conditions. Which set of fundamental solutions is chosen, also depends on the nature of the boundaries. It is therefore analogous to the method of solving the wave equation in acoustic problems by representing the solution as a series of orthogonal functions each satisfying the wave equation: in the one-dimensional case, as a Fourier Series. The fundamental difficulty of this method is the problem of obtaining the optimum set of fundamental solutions which will satisfy the boundary conditions. In the present problem of diffusion between parallel planes, the set has been found. Certain of the results given below have already been determined by this method by Rounds (1955). (Indication of these solutions will be given in the appropriate section). In order to demonstrate this method, the first of the two problems in V is treated in this way.

The alternative approach, being the one otherwise used here, is that of Heaviside Operational calculus. This method has been used successfully in other diffusion problems, its chief merit being that it rids the problem of any kind of apparent arbitrariness in the determination of the solution (for example, the difficulty mentioned above in Sutton's approach). In the Heaviside method, the solution in terms of x and z is related to the solution in operational form by the well-known integral relationship. When exact solutions are not to be found by either method it is usually true that this is because in the Heaviside method, the operational solution has one or more branch points. The occurrence of these branch points is not always an

unsurmountable difficulty but generally one is forced to be satisfied with asymptotic solutions. Goldstein's interpretation formulae (Jeffreys) are often an invaluable aid in dealing with branch points and several more complex forms have been solved, apart from the usual standard results. At any rate, the fact that asymptotic solutions can invariably be found by this method appears to be an obvious advantage over the other method where no such opportunities present themselves. In the present problem of the diffusion between parallel planes the above difficulty will, in the simpler cases, not arise; the operational solutions have poles but no branch points.

The restriction to obtaining a complete solution for every occasion arises from the necessity of being able to represent $K(z)$ by a simple function of z. If sufficient simplicity is achieved, solutions in the operational form can be found and the boundary conditions applied. When the co-ordinates are as shown in figure (i), solutions have been found with:

$$\text{(i)} \quad K = \text{constant.}$$

$$\text{(ii)} \quad K = kz^{\alpha} \qquad 0 \leqslant z \leqslant H$$

$$\text{(iii)} \quad K = k(H - z)^{\alpha} \qquad 0 \leqslant z \leqslant H$$

and in two further cases. Generally it can be assumed that under such conditions as we are trying to represent, the eddy diffusivity would increase with height from zero at the ground to some maximum before falling away to zero again at the base of the stable layer. The following models, representative of this behaviour, have been solved

$$\text{(iv)} \quad K = kz(H - z) \qquad 0 \leqslant z \leqslant H$$

$$\text{(v)} \quad \begin{aligned} K &= kz & 0 &\leqslant z \leqslant \tfrac{1}{2}H \\ &= k(H - z) & \tfrac{1}{2}H &\leqslant z \leqslant H \end{aligned}$$

Since the source can be at any height, it is clear that solutions (ii) and (iii) are mathematically identical. Consequently only the solution of (iii) will be given in any detail. Details of the solutions of (i), (iii), are given in III and IV respectively, (iv) and (v) in V and the result only of (ii) in IV.

II. INTRODUCTION TO THE CONVECTION-DIFFUSION MODEL

The diffusion of the emittant is aided by the presence of free thermal convection. Prolonged heating of the ground by solar radiation results in very complex motion of the air in the adjacent layer. Precisely what this behaviour is, is still not fully understood, but very broadly the picture would seem to be this: in the lowest layer of the air (depth of the order of 100 ft.), the air is very unstable and mechanical turbulence, arising from the break down of the velocity shear and the surface roughness, together with forced convection, consisting of small irregular bubbles and jets of hot air rising from thermal sources at the ground, results in a random but quasi-continuous flux of heat away from the surface. The energy spectrum of this turbulence will have its largest components in the moderately high wave numbers.

Above this region is a more stable layer, perhaps several thousand feet thick, with nearly zero lapse conditions, in which there will be very little mechanical turbulence. The transitional zone between these two layers appears to be fairly thin and for some purposes might be considered to be a surface of discontinuity. As this inter-surface is unstable to large perturbations received from below, a succession of discrete bubbles will break away from the lower layer and rise through the quasi-adiabatic layer. In its ascent a bubble will entrain air from the surrounding atmosphere and will consequently grow in size and slow down. In consequence, the magnitude of the energy spectrum of

the velocity perturbations in this upper layer is very small at the high frequency end, whereas the larger low frequency components will correspond to the motion resulting from the penetrative convection process. Recordings of temperature fluctuations (James 1953) confirm this. This picture is based on the work by Scorer (1954). It is also confirmed by Priestley's work (1955) in which he shows that for small negative values of the Richardson number (Ri) (i.e. near the ground, since (Ri) increases with height) the vertical flux of heat depends on the Richardson number and thus on the mechanical turbulence and forced convection, whereas for larger values of (Ri) (i.e. at larger heights) a relatively sharp transition takes place to the state in which the flux of heat is independent of Ri. This may be interpreted as flux by free convection. Similar confirmation comes from other workers.

The model presented in this paper is a first attempt to represent the main effects of the diffusion-convection processes on the dispersal of an emittant. In the future, it is hoped to improve the model so that it may be more consistent with the physical realities. In the present model, we are concerned with dispersal in a layer of finite depth; that is, the stability increases above the quasi-adiabatic layer so rapidly that no penetration occurs through the boundary. Also we shall neglect any effects due to the condensation of water vapour. Our model, which is designed to give statistical representation of the phenomena rather than to deal with the fine detail of the process, is then this: the dispersion may be separated into two distinct processes, firstly that of mechanical turbulence and forced convection which we shall assume can be represented by a coefficient of eddy diffusivity $K(z)$, (this will be large near the ground and will fall off to zero at the top of the layer), and secondly, that of free convection consisting of a succession of discrete bubbles typical of conditions at the ground rising to the top of the layer where they diffuse out mixing with the local atmosphere. The concept of discrete bubbles is not universally accepted as the complete representation of the free convection process although it has much support. However it is not essential to the model since this is to be a statistical approach, but on the other hand it is convenient to the description. As these bubbles rise they may slow down and they may grow laterally but it is to be assumed that no intermixing takes place between the bubbles and the surrounding atmosphere. This is clearly one of the points in which an improvement is to be sought in any future model. The convection process, in effect, amounts to a sink on the ground, $z = 0$ (to account for the disappearance of matter carried away by the bubbles) and an equivalent source at the top of the layer (to account for the appearance of this matter carried up to the inversion by the bubbles), together with a continuous subsidence throughout the layer to balance the effect of the transfer (air as well as the emittant is transfered). The effect is most pronounced when the source is at the ground or very close to it. We shall therefore put $h = 0$: the source at ground level. Then the source-centred plume will neither be as dense nor as deep as without convection, on account of the sink action on $z = 0$. At the top of the layer, the convected emittant which has been removed from the source-centred plume, will remain fairly concentrated at first; the diffusive effect will initially be small on account of the increase of stability as the stable layer is reached. However as the emittant subsides with the arrival of fresh bubbles, it is increasingly influenced by diffusion, it spreads out more rapidly and eventually recombines with the plume spreading up from the ground.

The nature of the eddy diffusivity coefficient has already been discussed. We shall take $K(z)$ to decrease linearly to zero $K = K_o (H - z)$ except that, in order to avoid mathematical difficulties arising from the unrealistic discontinuity in dK/dz at $z = H$ (the inter-surface), we shall assume that very

near to this inter-surface, K departs from the linear law and goes smoothly to zero (i.e. dK/dz continuous across z = H) (see figure (ii)). Alternatively it may reasonably be supposed that the bubbles slightly overshoot the inter-surface due to their upward momentum and mix in a very thin layer just above the inter-surface. Mathematically, this produces the same result. This representation for K(z) does not fully take into account the rapid change in turbulence across the transitional zone described above, but it seems to be otherwise satisfactory. The equation of dispersion, representing the expectation concentration of the emittant, is derived in VI and is

$$u_o \frac{\partial C}{\partial x} = K_o \frac{\partial}{\partial z} (H - z) \frac{\partial C}{\partial z} + \lambda \frac{\partial C}{\partial z} \qquad (1)$$

where λ is a measure of the flux of matter from the ground to the inter-surface by convection and is most easily defined as the average velocity of subsidence of the atmosphere needed to balance the sink-source action of the convection process. λ is, under the assumptions made above, constant with height, and when the convection is at all appreciable it will be of the order of 1 cm/sec. or more. For such values of λ, the comparison between the equivalent problems, one with convection, the other without, shows very markedly the effect of convection. In the solution the strength of the convection is represented by the parameter $\beta = \lambda/K_o$. Now if H is about $0(10^5)$ and K at the ground, enhanced by forced convection, is about $0(10^5)$, then $K_o = \frac{K}{H} = 0(1)$ and thus β will also be 0(1). In fact the comparison is made for $\beta = \frac{1}{2}$, since this makes the computation particularly simple. The result is that the convection aids the diffusion process to such an extent that the emittant is, to all intents and purposes, uniformly distributed across the layer at a distance much less than half that in which uniformity is achieved without convection. This is clearly demonstrated in the corresponding figures (vii) and (viii).

This is indeed fortunate because a complete solution has not been obtained, only an asymptotic solution. But this approximation is so good that it holds throughout the whole region of interest, as these solutions confirm.

Two small corrections may be applied to the result if desired. In fact, if they were applied to figure (viii) very little difference would be observed at all; but for very small x they would be of importance. Firstly, the bubbles take a finite time to rise. Their terminal velocity v is presumably of the order of metres/sec. This means that the first term $\tilde{C}(x,z)$ (see VI), which represents the contribution from the convected emittant, should be 'displaced' that is, it should be replaced by a term $\tilde{C}(x - \frac{Hu_o}{v}, z)\, H(x - \frac{Hu_o}{v})$. This displacement is $0(10^{-3})$ compared with X_o (the distance in which uniformity is achieved). Secondly any recording instrument measuring the concentration of the emittant will not only record the concentration due to the surrounding subsiding atmosphere, but also occasionally it will record values corresponding to bubbles which are rising past it. Near the source this may well be important. Now the statistical proportion of the atmosphere filled with bubbles is $\frac{\lambda}{v}$ of the whole and at any position (x,z) in the field, the bubbles passing the point will have originated from the ground at a position $x_o = x - \frac{u_o z}{v}$. Thus to correct for this effect we can add on an extra term $\frac{\lambda}{v} C_o (x - \frac{u_o z}{v})$ where C_o is the ground level concentration (see figure (iii)).

III. DIFFUSION MODEL WITH CONSTANT DIFFUSIVITY

Consider case (i): $u = 1, K = 1, H = 1, h = 0$ (2)

obtained from the general case by making the transformation:

$$z \longrightarrow \bar{z}H, \quad x \longrightarrow \frac{\bar{x}u_oH}{K}, \qquad Q \longrightarrow u_oH\ \bar{Q}$$

Now drop the bars. The equation of diffusion is

$$\frac{\partial C}{\partial x} = \frac{\partial^2 C}{\partial z^2} \tag{3}$$

with boundary conditions $\frac{\partial C}{\partial z} = 0$ on $z = 1$

$$\int_0^1 C\,dz = Q \qquad \text{independent of } x. \tag{4}$$

In operational form (3) becomes

$$\frac{\partial^2 C}{\partial z^2} - pC = -pQ\ \delta(z) \tag{5}$$

the solution of which is

$$C = A(p)\, e^{\sqrt{p}z} + B(p)\, e^{-\sqrt{p}z} \tag{6}$$

where A(p) and B(p) are determined by (4):

$$B(p) = A(p)\, e^{2\sqrt{p}}$$

$$B(p) = A(p) + Q\sqrt{p}$$

$$\text{Thus } C(p,z) = \frac{Q\sqrt{p}}{e^{\sqrt{p}} - e^{-\sqrt{p}}} \left\{ e^{-\sqrt{p}(1-z)} + e^{+\sqrt{p}(1-z)} \right\} \tag{7}$$

$$\text{i.e. } C(x,z) = \frac{Q}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{\cosh \sqrt{p}(1-z)}{\sqrt{p}\ \sinh \sqrt{p}}\ e^{px}\,dp \tag{8}$$

This is an even function of $\sqrt{p}$. Therefore there are no branch points. The contributions come from the residues at the poles (i.e. the zeros of $\sqrt{p}\sinh\sqrt{p}$) These are at $p = -n^2\pi^2$ ($n = 1,2,3, \ldots\ldots\ldots$) and at $p = 0$.

The residue at $p = 0$ is Q; at $-n^2\pi^2$ it is $2Q(-1)^n \cos n\pi(1-z)\ e^{-n^2\pi^2 x}$

$$\text{Thus } C(x,z) = Q \left\{ 1 + 2 \sum_{n=1}^{\infty} (-1)^n \cos n\pi(1-z)\ e^{-n^2\pi^2 x} \right\} \tag{9}$$

valid for all x, z.
It is worth noting that this is very rapidly convergent except for very small x.

IV. DIFFUSION MODEL WITH $K = K_o\,(H - z)^{\alpha}$: ELEVATED LINE SOURCE

This example refers to case (iii) of the introduction. By obvious transformation of x and z, take $u = 1$
$K = (H - z)^{\alpha}$
$h = 1$.

In operational form the diffusion equation is

$$\frac{\partial}{\partial z}\left\{(H-z)^{\alpha}\ \frac{\partial C}{\partial z}\right\} - pC = -pQ\ \delta(z-1) \tag{10}$$

Put $H - z = t$, $n = \frac{1-\alpha}{2-\alpha}$, $r = \frac{2}{2-\alpha}\ \sqrt{p}\, t^{1-\frac{\alpha}{2}}$ (11)

Then the solution of (10) is

$$C_1 = A(p)\ r^n I_n(r) + D(p)\ r^n I_{-n}(r), \text{ in (1) (i.e. } 0 \leqslant z \leqslant 1) \tag{12}$$

$$C_2 = B(p)\ r^n I_{-n}(r), \text{ in (2): } 1 \leqslant z \leqslant H$$

(satisfying the boundary condition at $z = H$, $K\frac{\partial C}{\partial z} = 0$)

On $z = 0$; let $r = \rho$. Then since $K\frac{\partial C}{\partial z} = 0$ there, we have

$$A\ I_{n-1}(\rho) = -D\ I_{1-n}(\rho) \tag{13}$$

Also on $z = 1$ we have $C_1 = C_2$. Let $r = q$ on $z = 1$, then

$$A\ I_n(q) + D\ I_{-n}(q) = B\ I_{-n}(q) \tag{14}$$

Finally, we must apply the flux condition $Q = \int_0^H C\, dz$

$$Q = \frac{1}{p}\ \left(\frac{2-\alpha}{2\sqrt{p}}\right)^{1-2n}\ \left\{\int_0^q B(p)\ r^{1-n} I_{-n}(r)\, dr + \int_q^{\rho}\left\{A(p)\ r^{1-n}\ I_n(r) + D(p)\ r^{1-n}\ I_{-n}(r)\right\}dr\right. \tag{15}$$

which by (13), (14) and the Wronskian relationship reduces to

$$Q = \frac{1}{p}\ \left(\frac{2\sqrt{p}}{2-\alpha}\right)^{n}\ \frac{A(p)}{I_{-n}(q)}\ \frac{2}{\pi q}\ \sin n\pi \tag{16}$$

$$\therefore A(p) = -\frac{\pi}{2}\, Q\sqrt{p}\ \left(\frac{2\sqrt{p}}{2-\alpha}\right)^{1-n}\ \text{cosec}\ n\pi\ \ I_{-n}(q)$$

$$D(p) = \frac{\pi}{2}\, Q\ \text{cosec}\ n\pi\ \left(\frac{2\sqrt{p}}{2-\alpha}\right)^{1-n}\ \sqrt{p}\ I_{-n}(q)\ \frac{I_{n-1}(\rho)}{I_{1-n}(\rho)}$$

and $$B(p) = \frac{\pi}{2}\, Q\ \text{cosec}\ n\pi\ \left(\frac{2\sqrt{p}}{2-\alpha}\right)^{1-n}\sqrt{p}\ \left\{I_{-n}(q)\ \frac{I_{n-1}(\rho)}{I_{1-n}(\rho)} - I_n(q)\right\} \tag{17}$$

Consider $C_1 = A(p)\ r^n\ I_n(r) + D(p)\ r^n\ I_{-n}(r)$

The first term has no poles or branch points, and therefore has no contribution. The second term has no branch points but gives a contribution from the residues at the poles.

For both regions (1) and (2)

$$C = \frac{\pi}{2-\alpha} \; Q \operatorname{cosec} n\pi \; t^{\frac{1-\alpha}{2}} \; p \; I_{-n}(q) \; \frac{I_{n-1}(\rho)}{I_{1-n}(\rho)} \; I_{-n}(r) \qquad (18)$$

At $p = 0$ the contribution is $\frac{Q}{H}$

If s_i are the roots of $J_{1-n}(s) = 0$ then (18) has poles at $p = -s_i^2/a^2$

where $a = \frac{2}{2-\alpha} H^{1-\frac{\alpha}{2}}$. The sum of the residues gives the result:

$$C = \frac{Q}{H} \left\{ 1 + \frac{2-\alpha}{2} \pi \operatorname{cosec} n\pi \frac{(H-z)^{\frac{1-\alpha}{2}}}{H^{1-\alpha}} \sum_{i=1}^{\infty} s_i \frac{J_{-n}(H^{\frac{\alpha-2}{2}} s_i) J_{n-1}(s_i) J_{-n}((\frac{t}{H})^{\frac{2-\alpha}{2}} s_i)}{J_{-n}(s_i)} \; e^{-s_i^2 x/a^2} \right\} \qquad (19)$$

These solutions have been obtained by Rounds using Sutton's method.
In the special case of $h = 0$, $H = 1$, $K = 1 - z$, equation (19) reduces to the following form:

$$C = Q \left\{ 1 + \sum_{i=1}^{\infty} \frac{J_0(s_i\sqrt{t})}{J_0(s_i)} \; e^{-s_i^2 x/4} \right\} \qquad (20)$$

where s_i are the zeros of $J_1(s)$

This is the solution which will be used in the comparison in VII.
The more general form than this when $h = 0$, $H = 1$, $K = (1 - z)^{\alpha}$ is

$$C = Q \left\{ 1 + (2-\alpha) t^{\frac{1-\alpha}{2}} \sum_{i=1}^{\infty} \frac{J_{-n}(s_i t^{1-\frac{\alpha}{2}})}{J_{-n}(s_i)} \; e^{-s_i^2 x/a^2} \right\} \qquad (21)$$

where $a = \frac{2}{2-\alpha}$, s_i are the zeros of $J_{1-n}(s)$

The solution to problem (ii) of the introduction: $u = 1$, $K = z^{\alpha}$, $h = 0$, $H = 1$; can be deduced from (19).

It is $$C = Q \left\{ 1 + (n-1)! \; z^{\frac{1-\alpha}{2}} \sum_{i=1}^{\infty} s_i^{1-n} \frac{J_{n-1}(s_i) J_{-n}(z^{1-\frac{\alpha}{2}} s_i)}{J_{-n}(s_i)} \; e^{-s_1^2 x/a^2} \right\} \qquad (22)$$

In the special case of $\alpha = 1$, $n = 0$ we obtain the solution by a limiting process This gives

$$C = Q \left\{ 1 - \frac{\pi}{2} \sum_{i=1}^{\infty} s_i \frac{Y_1(s_i) J_0(s_i\sqrt{z})}{J_0(s_i)} \; e^{-s_i^2 x/4} \right\} \qquad (23)$$

All these series are rapidly convergent and the Bessel functions are tabulated for a selection of values of n in Watson's "Theory of Bessel Functions".
The result (20) is shown graphically in the comparison (figure (vii)).

V. THREE-REGION DIFFUSION MODEL (CASE (v)) AND CASE (iv): $K = z(1 - z)$

This section deals with cases in which K is zero at both the ground ($z = 0$) and the upper surface.

Firstly, case (iv) of the introduction, with the source elevated at height h and

$$\begin{aligned} u &= 1 \\ K &= z(1 - z) \end{aligned} \tag{24}$$

For purposes of illustration we shall treat this problem using Sutton's approach, although it may equally be done using operational methods.

The equation of diffusion is

$$\frac{\partial C}{\partial x} = \frac{\partial}{\partial z} \left\{ z(1-z) \frac{\partial C}{\partial z} \right\} \tag{25}$$

Separate the variables. Put $C = X(x)\, Z(z)$

Then $X(x) = A\, e^{-\sigma x}$ (26)

where σ is some constant to be determined.

$Z(z)$ satisfied $\frac{\partial}{\partial z} \left(z(1-z) \frac{\partial Z}{\partial z} \right) + \sigma Z = 0$ (27)

Solutions satisfying the boundary conditions on $z = 0$ and $z = 1$ are obtained by putting $\sigma = n(n + 1)$ with n integer, when the solutions are Legendre Polynomials:

$$C = A(n)\, e^{-n(n+1)x}\, P_n(y) \qquad \text{when } y = 2z - 1 \tag{28}$$

Now on $x = 0$: $C = Q\, \delta(z-h)$ in delta function notation

$$\therefore \sum_{n=0}^{\infty} A(n)\, P_n(y) = Q\, \delta(z-h) \tag{29}$$

Multiply both sides by $P_m(y)$ and integrate with respect to y between (-1, 1)

$$\begin{aligned} \frac{2}{2m+1} A(m) &= \int_{-1}^{1} P_m(y)\, Q\, \delta(\tfrac{1}{2}(1+y)-h)\, dy \\ &= 2Q\, P_m(2h-1) \end{aligned} \tag{30}$$

Thus the solution is

$$C = Q \left\{ 1 + \sum_{n=1}^{\infty} (2n+1)\, P_n(2h-1)\, P_n(2z-1)\, e^{-n(n+1)x} \right\} \tag{31}$$

where the first few of the Legendre Polynomials are

$$P_0(z) = 1,\ P_1(z) = z,\ P_2(z) = \tfrac{1}{2}(3z^2 - 1),\ P_3(z) = \tfrac{1}{2}(5z^3 - 3z).$$

The second example, namely case (v) of the introduction is the three-region diffusion model. Transform the variables (as in IV) so that

$$u = 1$$
$$K = z \text{ for } 0 \leqslant z \leqslant 1 \quad \text{(region 1)}$$
$$= 2 - z \text{ for } 1 \leqslant z \leqslant 2 \quad \text{(region 2)} \qquad (32)$$
$$= 0 \text{ for } z \leqslant 2$$
(see figure (v))

In the upper region put $2 - z = t$.

Then the solution in operation form, valid in this region, and satisfying the boundary conditions of zero flux at $z = 2$ is

$$C_2 = B(p)\, I_0(2\sqrt{p}t) \qquad (33)$$

In the lower region the solution is

$$C_1 = E(p)\, I_0(2\sqrt{p}z) + D(p)\, K_0(2\sqrt{p}z) \qquad (34)$$

The three unknown functions of p are determined from the boundary conditions

(a) $C_1 = C_2$ on $z = 1$

(b) $\frac{\partial C_1}{\partial z} = \frac{\partial C_2}{\partial z}$ on $z = 1$

(c) Flux condition $Q = \int_0^1 C_1\, dz + \int_0^1 C_2\, dt$.

By (c) we deduce $D = 2pQ$, and by the Wronskian relationship, (a) and (b) give

$$B = p\,\frac{Q}{2}\,\frac{1}{I_0(2\sqrt{p})I_1(2\sqrt{p})}, \quad E = Qp\,\frac{K_1(2\sqrt{p})I_0(2\sqrt{p}) - K_0(2\sqrt{p})I_1(2\sqrt{p})}{I_0(2\sqrt{p})I_1(2\sqrt{p})}$$

Thus the solution in the upper region is

$$C_2(x,z) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{Q\, I_0(2\sqrt{p}t)}{2\sqrt{p}I_0(2\sqrt{p})I_1(2\sqrt{p})}\, e^{px}\, dp \qquad (35)$$

The integrand is an even function of $\sqrt{p}$ and therefore has no branch points. Its value comes from the residues at the zeros of $I_1(2\sqrt{p})$ and $I_0(2\sqrt{p})$. If s_i is a root of $J_1(s) = 0$ and r_i a root of $J_0(r) = 0$ then we obtain

$$C_2(x,z) = \frac{Q}{2}\left\{1 + \sum_{i=1}^{\infty} \frac{J_0(\sqrt{2-z}\,s_i)}{J_0^2(s_i)}\, e^{-s_i^2 x/4} - \sum_{i=1}^{\infty} \frac{J_0(\sqrt{2-z}\,r_i)}{J_1^2(r_i)}\, e^{-r_i^2 x/4}\right\} \qquad (36)$$

In the lower region

$$C_1(p,z) = \frac{Q/\sqrt{p}\; I_0(2\sqrt{pz})}{2I_0(2\sqrt{p})I_1(2\sqrt{p})} - 2Qp\,\frac{K_0(2\sqrt{p})}{I_0(2\sqrt{p})}\, I_0(2\sqrt{pz}) + 2Qp\, K_0(2\sqrt{pz}) \qquad (37)$$

The first term is identical to $C_2(p,z)$ and has the same interpretation. The interpretation of the other two terms is best obtained by a limiting process

$$p\left\{-\frac{K_o(a\sqrt{p})}{I_o(a\sqrt{p})}\,I_o(b\sqrt{p}) + K_o(b\sqrt{p})\right\}$$

$$= \lim_{n\to 0}\left\{-\frac{K_n(a\sqrt{p})}{I_{-n}(a\sqrt{p})}\,I_{-n}(b\sqrt{p}) + K_n(b\sqrt{p})\right\} p^{1-\frac{n}{2}}$$

$$= \lim_{n\to 0} p^{1-\frac{n}{2}}\,\frac{\pi}{2}\,\mathrm{cosec}\,n\pi\left\{\frac{I_n(a\sqrt{p})}{I_{-n}(a\sqrt{p})}\,I_{-n}(b\sqrt{p}) - I_n(b\sqrt{p})\right\} \qquad (38)$$

The second of these terms has neither branch points nor poles. The first term has poles, with residues

$$= \frac{\pi}{2}\,\mathrm{cosec}\,n\pi\,\frac{2}{a}\left(\frac{r_i}{a}\right)^{1-n}\frac{J_n(r_i)J_{-n}\left(\frac{b}{a}r_i\right)}{J_{1-n}(r_i)}\,e^{-r_i^2x/a^2}$$

at $p = -r_i^2/a^2$ where r_i is the i^{th} zero of $J_{-n}(r)$

As $n \longrightarrow 0$ we have $\mathrm{cosec}\,n\pi\;J_n(r_i) = Y_o(r_i)$

Hence the interpretation of equation (37) is

$$C_1(x,z) = \frac{Q}{2}\left\{1 + \sum_{i=1}^{\infty}\frac{J_o(\sqrt{z}\,s_i)}{J_o^2(s_i)}\,e^{-s_i^2x/4} + \sum_{i=1}^{\infty}\left\{\pi r_i\,\frac{Y_o(r_i)}{J_1(r_i)}\,J_o(\sqrt{z}\,r_i) - \frac{J_o(\sqrt{z}\,r_i)}{J_1^2(r_i)}\right\}e^{-r_i^2x/4}\right\} \qquad (39)$$

It is clear that $C_1 = C_2$ on $z = 1$, as indeed it must do, since $J_o(r_i) = 0$. The extension of this model to more general initial data seems to be extremely difficult, except possibly in a few cases, on account of the introduction of branch points into the operational solutions.

VI. DIFFUSION AIDED BY CONVECTION

The theoretical aspects of this model have been discussed in II. It remains to specify the boundary conditions more precisely and to solve the mathematical equations.

Firstly, we must consider the boundary condition at the top of the layer. Considering a small element ABCD (see figure (vi)) at the inter-surface, then, with assumptions made, the rate of change of concentration in the element is

$$\frac{\partial C}{\partial t}\,dx\,dz = 0 = u_oC(x)dz - u_oC(x+dx)dz + \lambda\,dz\,dx\,(C_o(x) - C(x))$$

where $C(x)$ is the ground level concentration of which the bubbles are typical.

λ is the velocity of subsidence.

Hence $$u_o \frac{\partial C}{\partial x} = \lambda \left\{ C_o(x) - C(x) \right\} \qquad (40)$$

We are neglecting the refinements suggested in II. If required, the necessary alterations are easily made.

The equation of diffusion is

$$u_o \frac{\partial C}{\partial x} = K_o \frac{\partial}{\partial z} (H - z) \frac{\partial C}{\partial z} + \lambda \frac{\partial C}{\partial z} \qquad (41)$$

Put $z = H\bar{z}$, $1 - \bar{z} = t$, $x = \frac{u_o H}{K_o} \bar{x}$, $\frac{\lambda}{K_o} = \beta$ (42)

where $K_o = 0$ (1)

Dropping the bar on $\bar{x}$ we obtain

$$\frac{\partial C}{\partial x} = \frac{\partial}{\partial t} (t \frac{\partial C}{\partial t}) - \beta \frac{\partial C}{\partial t} \qquad (43)$$

with boundary conditions

$$\frac{\partial C}{\partial x} = \beta H \left\{ C_o(x) - C(x) \right\} \text{ on } t = 0$$

$$\frac{Q}{u_o H} = \int_o^1 C \, dt \qquad (44)$$

Putting (43) into operational form, the solution is

$$C = A(p) \, r^\beta \, K_\beta(r) + B(p) \, r^\beta \, I_\beta(r) \qquad (45)$$

where $2\sqrt{pt} = r$

Applying the boundary conditions (44):

$$\text{On } r = 0: \quad A(p) \frac{\pi}{2} \frac{\operatorname{cosec} \beta\pi}{(-\beta)!} 2^\beta = \frac{\beta H}{p+\beta H} \left\{ A(p) \, (2\sqrt{p})^\beta K_\beta(2\sqrt{p}) + B(p) \, (2\sqrt{p})^\beta I_\beta(2\sqrt{p}) \right\} \qquad (46)$$

Now βH is a large quantity and except for very small x (large p) it will dominate the denominator on the right-hand side. When x is other than very small, the solution is of most interest: this will be represented by solutions (a). Solutions (b) and (c) refer to very small x and to the solution on r = 0 for all x, respectively.

The flux condition gives, for $x < 1$, large p:

$$\text{(a):} \quad 2p \frac{Q}{u_o H} = \frac{\Lambda}{\beta H} \frac{\pi}{2} \operatorname{cosec} (1 + \beta)\pi \frac{2^{1+\beta}}{(-1-\beta)!} p^{-\frac{1}{4}+\frac{\beta}{2}} e^{2\sqrt{p}} B(p) + B(p) \, (2\sqrt{p})^{1+\beta} \frac{e^{2\sqrt{p}}}{2\sqrt{\pi} p^{\frac{1}{4}}} \qquad (47)$$

where $\Lambda = \frac{\sin \beta\pi}{\pi^{3/2}} (-\beta)! \, \beta H$

and similarly for (b) and (c). Taking the second term, which is dominant, we obtain $B(p)$. Substitution into (46) gives $A(p)$.

$B(p)$ is independent of whether case (a), (b) or (c) is being considered.

Thus

$$C(p,z) = 2^{1-\beta}\frac{Q}{u_oH}\sqrt{\pi}\,p^{\frac{3}{4}-\frac{\beta}{2}}\,e^{-2\sqrt{p}}r^{\beta}I_{\beta}(r) + \begin{cases} \frac{Q}{u_oH}\wedge\frac{\sqrt{\pi}}{\beta H}\,2^{1-\beta}\,p^{\frac{1}{2}}\,r^{\beta}K_{\beta}(r) \;\ldots\ldots\ldots & (a) \\ \frac{Q}{u_oH}\wedge 2^{1-\beta}\sqrt{\pi}\,p^{-\frac{1}{2}}\,r^{\beta}K_{\beta}(r) \;\ldots\ldots\ldots & (b) \\ \frac{Q}{u_oH}\wedge 2\sqrt{\pi}\,\frac{p^{\frac{1}{2}}}{p+\beta H}\,\frac{\pi}{2}\,\frac{\operatorname{cosec}\beta\pi}{(-\beta)!} \;\ldots\ldots & (c) \end{cases} \tag{48}$$

The first of these terms we denote by $\tilde{\tilde{C}}(p,z)$ and the second by $\tilde{C}(p,z)$. This solution is valid for moderately small x:

$$\tilde{\tilde{C}}(p,z) \backsimeq 2^{1-\beta}\frac{Q}{u_oH}\;(2\sqrt{t})^{\beta}\;2p\;K_{\beta}(2\sqrt{p})\;I_{\beta}(2\sqrt{t}\,\sqrt{p}) \quad \text{since p is large.}$$

Putting $2 = b$, $2\sqrt{t} = a$ the interpretation of $pK_{\beta}(2\sqrt{p})I_{\beta}(2\sqrt{pt})$ is

$$I = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} I_{\beta}(a\sqrt{p})\,K_{\beta}(b\sqrt{p})\,e^{px}\,dp \tag{49}$$

Following the method of § 2, (Smith 1957) we obtain, by deforming the contour to a loop round the negative real axis,

$$\tilde{\tilde{C}}(x,z) = 2\frac{Q}{u_oH}\;\frac{t^{\beta/2}}{x}\;e^{-\frac{1+t}{x}}\;I_{\beta}\left(\frac{2\sqrt{t}}{x}\right) \tag{50}$$

valid for all z and moderately small x.

$\tilde{\tilde{C}}(x,z)$ represents the concentration derived from the source-centred plume whereas $\tilde{C}(x,z)$ gives the concentration due to the emittant which has been convected to the top of the layer.

$C(p,z)$ (a) and (b) can be interpreted using Goldstein's interpretation formulae for Bessel Functions. $C(p,z)$ (c) with $r = 0$ is also a standard form given in "Tables of Integral Transforms": Bateman Manuscript Project, C.I.T.

The results can be summarized as follows:

$$\tilde{C}(x,z)\;(c) = 2\frac{Q}{u_oH}\wedge\frac{\sqrt{\pi}}{(-\beta)!}\;\frac{\pi}{2}\operatorname{cosec}\beta\pi\,\frac{1}{\beta H}\,f(\beta Hx) \tag{51}$$

where $f(S^2) = \frac{2}{\pi}\,e^{S^2}\int_0^S e^{t^2}\,dt$, tabulated in Jahnke-Emde

$$\tilde{C}(x,z)\;(b) = \frac{Q}{\sqrt{\pi}\,u_o}\;x^{\frac{1}{2}}\;e^{-t/x}\left\{{}_1F_1\left(\tfrac{3}{2}-\beta;\;1-\beta;\tfrac{t}{x}\right) - \frac{(-\tfrac{1}{2})!\,(-\beta)!}{(\tfrac{1}{2}-\beta)!\;\beta!}\;\left(\frac{t}{x}\right)^{\beta}\;{}_1F_1\left(\tfrac{3}{2};\;1+\beta;\tfrac{t}{x}\right)\right\} \tag{52}$$

valid for very small x.

$$\tilde{C}(x,z)\ (a) = \frac{Q}{u_o H(\beta-1)!}\ x^{-\frac{1}{2}}\ e^{t/x}\ U(\tfrac{1}{2}-\beta;\ 1-\beta;\ \frac{t}{x}) \tag{53}$$

It is equation (53) which is of primary importance, being valid over practically all the range of x and z in which we are interested.

The final solution is thus

$$C(x,z) = \tilde{C}(x,z) + \bar{\bar{C}}(x,z) \tag{54}$$

How effective and realistic these results are, is best shown by evaluating a specific example. The next section deals with a comparison between a diffusion model and a diffusion-convection model. It is seen that the picture provided by these formulae is consistent with what one would expect from the physical assumptions.

VII. EVALUATION AND COMPARISON

To determine the effect of convection on the diffusion process is most easily accomplished by comparing the results of the last section, with specific values of the parameter involved, with the results obtained by evaluating the corresponding simple-diffusion solution (20) of IV.

Inspection of these solutions shows that for purposes of comparison only one parameter has to be chosen. That is the convection parameter β. It has already been shown that β is of order one. Therefore, with an eye to simplifying the calculations as much as possible, we shall take $\beta = \frac{1}{2}$. This, as will be seen, gives results which bring out the effect of convection very clearly without swamping the natural diffusion processes.

Equation (54) becomes, with $\beta = \frac{1}{2}$,

$$\frac{u_o H}{Q} C = \frac{1}{\sqrt{\pi x}} \left\{ e^{-t/x} + e^{-\frac{1+t}{x}} \left\{ e^{\frac{2\sqrt{t}}{x}} - e^{-\frac{2\sqrt{t}}{x}} \right\} \right\} \tag{55}$$

Equation (20) becomes

$$\frac{u_o H}{Q} C = 1 + \sum_{i=1}^{\infty} \frac{J_o(s_i\sqrt{t})}{J_o(s_i)}\ e^{-s_i^2 x/4} \tag{56}$$

The results are shown graphically where they may best be compared. (See figures (vii) and (viii)). One important feature, which justifies the assumption, made in the previous section, about having x small, is that with convection, uniformity is abhieved in under half the distance that is required without the convection.

ACKNOWLEDGEMENTS

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TABLE OF NOTATION *

C	expectation concentration (statistical mean).
h	height of source above ground.
H	height of inversion base.
$K(z)$	eddy diffusivity coefficient.
k, K_o	constant multipliers involved in $K(z)$, e.g. $K(z) = kz(H - z)$.
Q	flux of emittant.
u, u_o	constant velocity in x direction.
v	velocity of ascent of the bubbles.
x	distance downstream.
z	height above ground.
λ	velocity of subsidence of the environment.
$\beta = \frac{\lambda}{K_o}$	convection parameter; a measure of the respective importance of convection to diffusion.
$H(\xi) = 1$ if $\xi > 0$ $= 0$ if $\xi < 0$	Heaviside step function.
${}_1F_1(a; b; \xi)$, $U(a; b; \xi)$	confluent hypergeometric functions.
$I_n(\xi)$, $K_n(\xi)$, $J_n(\xi)$, $Y_n(\xi)$	Bessel functions.
$P_n(z)$	Legendre polynomials.

* Other notation used only in a limited part of the text is defined on introduction.

FIGURES

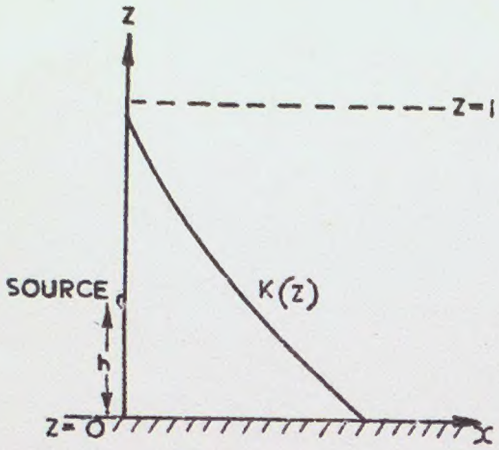
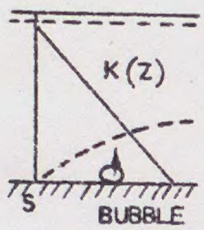
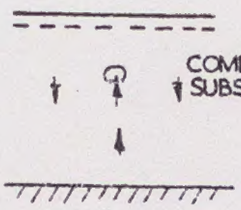
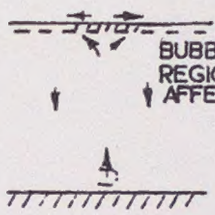


FIGURE (i)
THE DIFFUSION MODEL

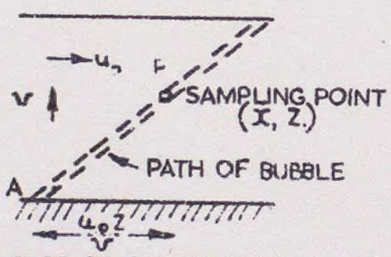


FIGURE(ii)

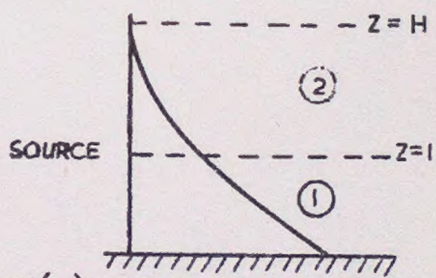




THE CONVECTION MODEL



FIGURE(iii) BUBBLES RISING FROM A ARE RECORDED IN THE 'CATCH' AT P



FIGURE(iv) REGIME FOR ELEVATED SOURCE WITH SIMPLE DIFFUSION

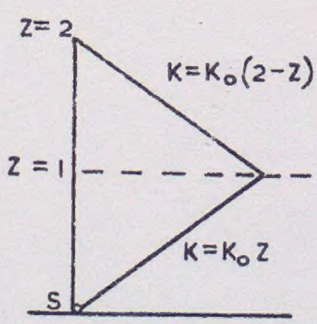


FIGURE (v)
THE THREE-REGION DIFFUSION MODEL

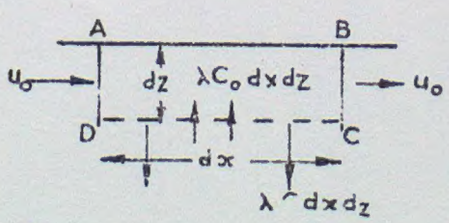


FIGURE (vi)

THE ELEMENT ABCD AT THE INTERSURFACE BETWEEN THE TURBULENT AND STABLE LAYERS.

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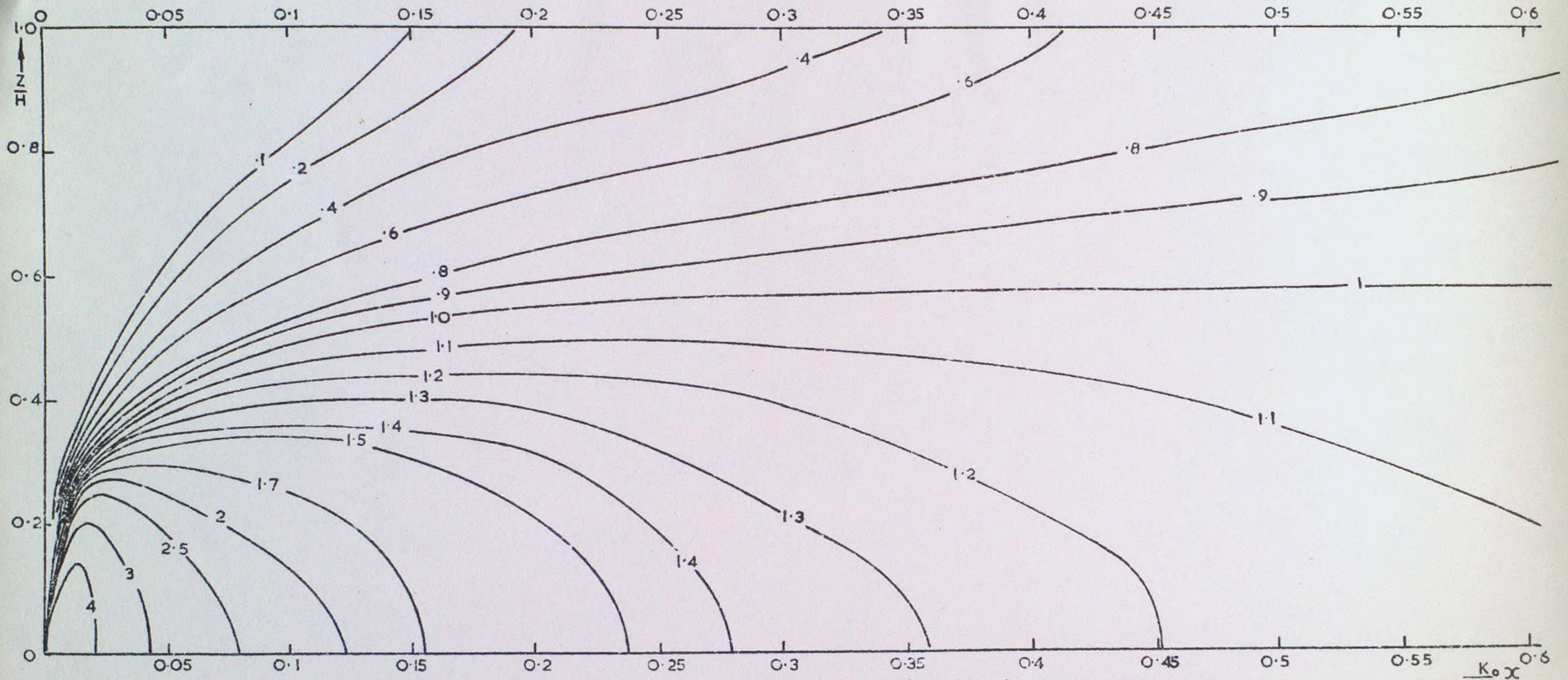
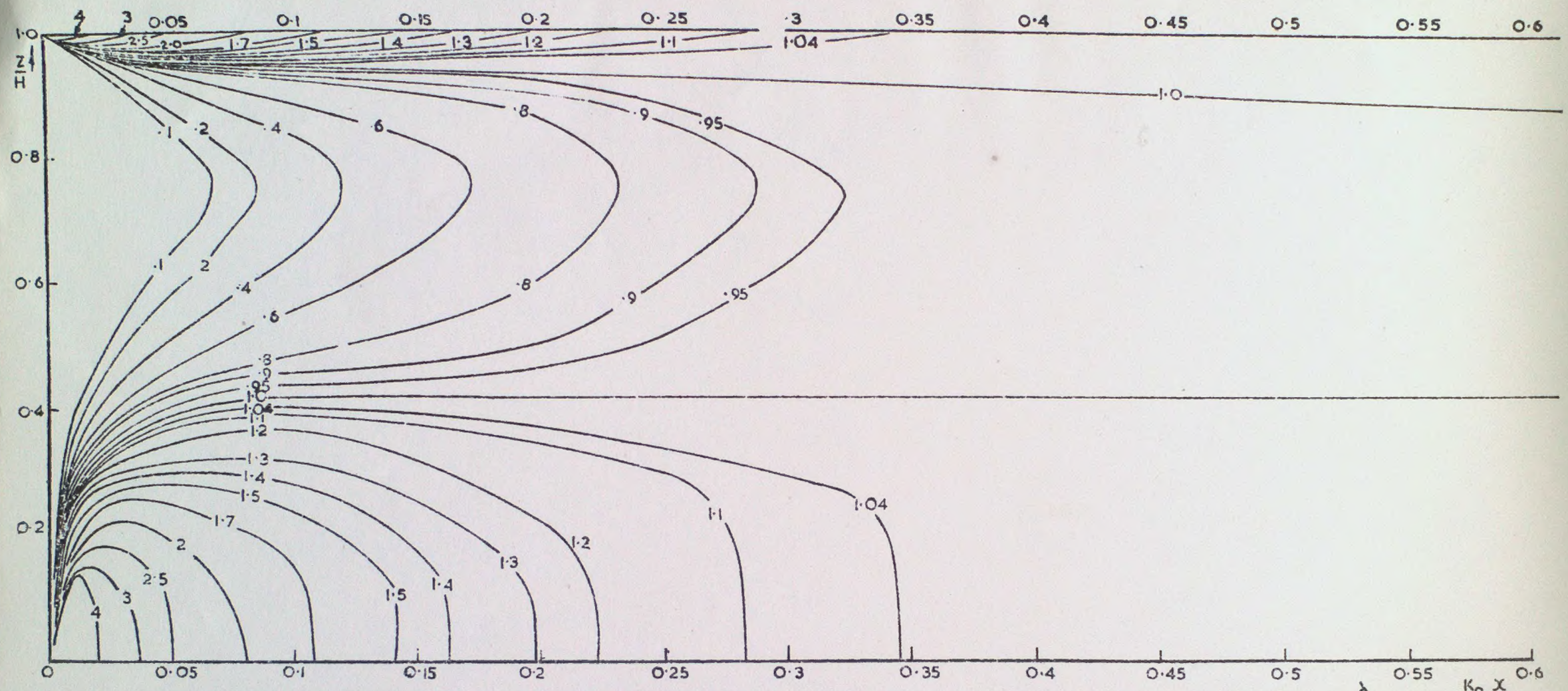


FIGURE (vii) LINES OF EQUAL CONCENTRATION. DIFFUSION MODEL, LINE SOURCE AT THE GROUND ($x = 0$, $z = 0$). THE VELOCITY IS CONSTANT ($u = u_0$) AND THE EDDY DIFFUSIVITY COEFFICIENT DECREASES LINEARLY WITH HEIGHT $K = K_0 (H - z)$. CONCENTRATION IS PLOTTED AS MULTIPLES OF THE FINAL UNIFORM CONCENTRATION $\frac{Q}{u_0 H}$



FIGURE(viii) LINES OF EQUAL CONCENTRATION. CONVECTION-AIDED DIFFUSION MODEL WITH CONDITIONS AS IN FIGURE(vii). CONVECTION PARAMETER $\beta = \frac{\lambda}{K_0}$ WHICH MEASURES THE RELATIVE IMPORTANCE OF CONVECTION TO DIFFUSION, IS TAKEN AS $\frac{1}{2}$ (λ IS THE VELOCITY OF COMPENSATING SUBSIDENCE).

THE CORRECTION TERM $\frac{\lambda}{v} C \left(x - \frac{uz}{v}\right)$, WHICH SHOULD BE INCLUDED IN THE SOLUTION SINCE A MEASURING INSTRUMENT WOULD PERIODICALLY BE TAKING SAMPLES FROM THE ASCENDING BUBBLES THEMSELVES, IS NOT INCLUDED IN THIS FIGURE. THIS IS BECAUSE NOT ONLY IN GENERAL WILL IT BE SMALL BUT ALSO IT DEPENDS ON A FURTHER PARAMETER v (THE VELOCITY OF ASCENT OF THE BUBBLE). THIS FIGURE, THEREFORE, GIVES THE MEAN CONCENTRATION OF THE ENVIRONMENT.